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Analysis of laminated circular cylinders of materials with the most general form of cylindrical anisotropy. II. Flexural deformations

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Abstract

This second paper of a two-part series is devoted to flexural stresses and deformations in a laminated anisotropic cylinder of perfectly bonded materials possessing the most general form of cylindrical anisotropy. The loading conditions include pure bending, flexure by a transverse force, and applied surface tractions that lead to resultant transverse loads and bending moments of uniform or linear variation along the axis. These loading conditions cause strain and stress fields that are uniform, linear and quadratically varying along the axis. The solutions herein are based on the relaxed formulation of the Saint-Venant's and Almansi–Michell problems where conditions on the ends of the cylinder are satisfied on an integral basis rather than on a point-wise basis. Differences in the stress distributions between these integral conditions with any point-wise specification are self-equilibrated states that decay with distance into the interior, i.e., Saint-Venant's principle. Means to account for such effects are discussed. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Circular cylinder; Anisotropy; Flexural elastostatics

1. Introduction

In the previous paper, Huang and Dong (2001) considered the analysis of axisymmetrical stresses and deformations in a laminated circular cylinder of perfectly bonded layers with the most general form of cylindrical anisotropy. The formulation was based on linear three-dimensional elasticity and semi-analytical finite elements using an analytical form for the axial dependence. Solutions according to a relaxed formulation were found using integral end conditions as opposed to their point-wise specification. Herein, an analogous analysis is given for flexural stresses and deformations of the same laminated anisotropic cylinder, where the loading condition is cast in a power series of the axial coordinate z. The first three load terms of this series are addressed herein and they are associated with de Saint-Venant (1856a,b) bending

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and flexure and the Almansi (1901a,b) and Michell (1901) problems. They relate to flexural behavior characterized by uniform, linearly varying and quadratically varying strains and stresses along the axis of the cylinder. A uniform state occurs for pure bending. Linearly varying stress states result from flexure by a transverse force on the end of the cylinder as well as from uniform axial longitudinal shear traction of sinusoidal circumferential variation to yield a resultant bending moment per unit length. Quadratically varying stress states are due to uniformly applied normal tractions and linearly varying longitudinal surface shear traction. Such loads possess resultant transverse load and bending moment gradients. From the discussion of these load cases, the solution procedure for the higher load terms will become clear. These solutions based on integral end conditions differ with those by a point-wise specification only in their distributions. The differences are self-equilibrated stress states which decay into the interior according to Saint-Venant's principle. Means for quantifying such differences are discussed.

The state-of-the-art survey in the previous paper revealed that the body of work on anisotropic cylinders was devoted almost entirely to axisymmetric deformations. The literature on flexural deformations by comparison showed a paucity of contributions. Lekhnitskii (1981) presented solutions for pure bending of a homogeneous cylindrically orthotropic cylinder and for flexure of a homogeneous cylindrically monotropic cylinder with the cross-sectional plane as an elastic symmetry plane. He remarked that the solution procedure for a fully anisotropic cylinder is relatively straight forward, but the formulas for stresses would be extremely complicated. The case of axial force, torsion and bending of a composite cylinder was considered by Kollar et al. (1992). Their analysis included applied surface tractions, but they restricted them to be uniform along the axis of the cylinder and free of a resultant axial force. Beyond these and to the best of the authors' knowledge, no other publications on this specific topic are available.

2. Equations for flexural deformations

We are concerned with a cylinder of length L that is composed of any number of bonded layers of materials with the most general form of cylindrical anisotropy. Set the origin of the cylindrical coordinate system (r,θ,z) at the center of the cross-section at the free or tip end and let z run positively toward the fixed end as shown in Fig. 1a. The governing equation for this cylinder is

$$\mathbf{K_1U} + \mathbf{K_2U}_{,\theta} + \mathbf{K_3U}_{,z} - \mathbf{K_4U}_{,\theta\theta} - \mathbf{K_5U}_{,\theta z} - \mathbf{K_6U}_{,zz} = \mathbf{F}, \tag{1}$$

where $\mathbf{U}(\theta,z)$ and $\mathbf{F}(\theta,z)$ are the assembled $(3N \times 1)$ displacement and load arrays of the N nodes in the finite element model and $\mathbf{K_i}$'s are system stiffness matrices. Details on the formulation of Eq. (1) can be found in Zhuang et al. (1999).

For flexural deformations, displacement vector $\mathbf{U}(\theta,z)$ and its counterpart $\mathbf{u}(\theta,z)$ on the element level consist of two sinusoidal components.

$$\mathbf{U}(\theta,z) = [\mathbf{U}_\mathrm{c}(z), \mathbf{U}_\mathrm{s}(z)]\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}, \qquad \mathbf{u}(\theta,z) = [\mathbf{u}_\mathrm{c}(z), \mathbf{u}_\mathrm{s}(z)]\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}, \tag{2}$$

where $\mathbf{U}_\mathrm{c}(z)$, $\mathbf{U}_\mathrm{s}(z)$ and $\mathbf{u}_\mathrm{c}(z)$, $\mathbf{u}_\mathrm{s}(z)$, contain the three nodal displacement components.

$$[\mathbf{U}_\mathrm{c}(z), \mathbf{U}_\mathrm{s}(z)] = \begin{bmatrix} U_{\mathrm{c}r}(z) & U_{\mathrm{s}r}(z)\\ U_{\mathrm{c}\theta}(z) & U_{\mathrm{s}\theta}(z)\\ U_{\mathrm{c}z}(z) & U_{\mathrm{s}z}(z)\end{bmatrix}, \qquad [\mathbf{u}_\mathrm{c}(z), \mathbf{u}_\mathrm{s}(z)] = \begin{bmatrix} u_{\mathrm{c}r}(z) & u_{\mathrm{s}r}(z)\\ u_{\mathrm{c}\theta}(z) & u_{\mathrm{s}\theta}(z)\\ u_{\mathrm{c}z}(z) & u_{\mathrm{s}z}(z)\end{bmatrix}. \tag{3}$$

Similarly, the load vector $\mathbf{F}(\theta,z)$ for applied tractions on the inner and outer surfaces of the cylinder can be written in terms of their sinusoidal components.

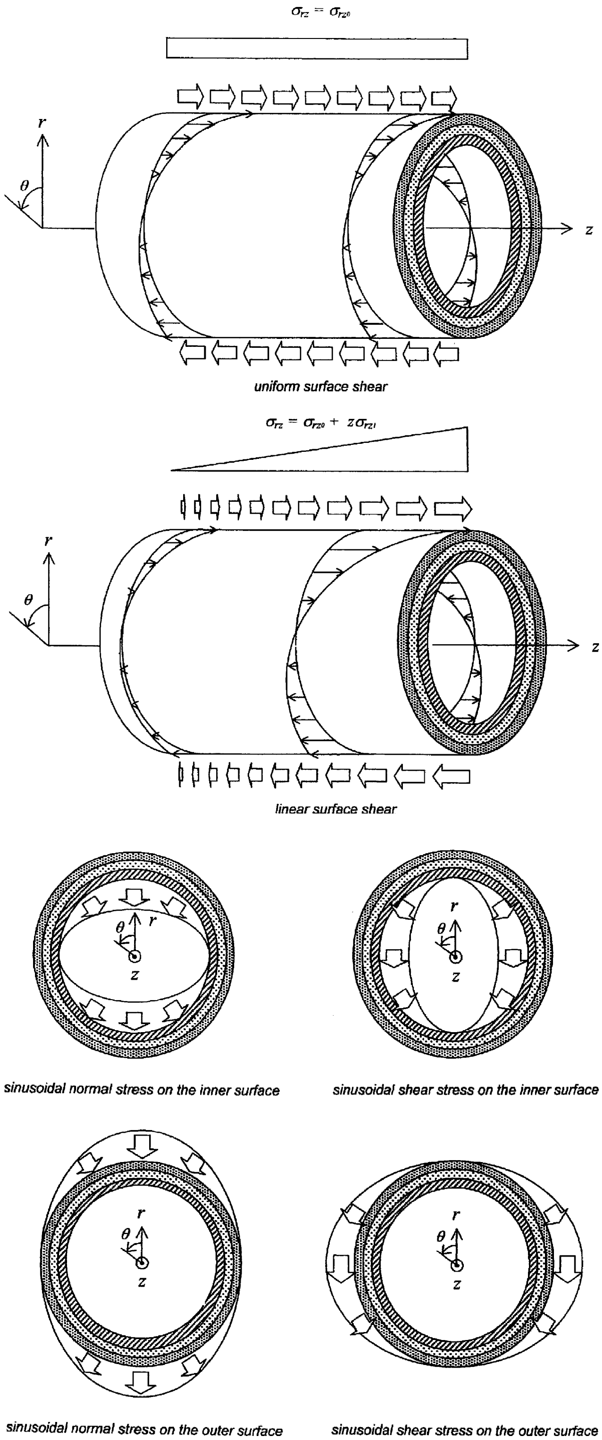

Fig. 1. (a) Circular cylinder under uniform and linearly varying longitudinal surface shears. (b) Circular cylinder under various normal and circumferential shear tractions.

$$\mathbf{F}(\theta, z) = [\mathbf{F}_c(z), \mathbf{F}_s(z)] \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}. \tag{4}$$

Although all three surface traction components may be included in $\mathbf{F}_c$ and $\mathbf{F}_s$, only the longitudinal surface shear tractions are represented in our solution scheme. These tractions have resultant bending moments when they are integrated over the surfaces where the tractions are applied. Normal and circumferential surface shear tractions leading to resultant transverse loads are accounted for directly in the displacements. Thus, $\mathbf{F}_c$ and $\mathbf{F}_s$ have the following forms.

$$[\mathbf{F}_c(z), \mathbf{F}_s(z)] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ F_{cz}(z) & F_{sz}(z) \end{bmatrix}. \tag{5}$$

Substitution of $\mathbf{U}$ and $\mathbf{F}$ from Eqs. (2) and (4) into Eq. (1) leads to an equation with two sinusoidal components, each component of which must vanish identically. These two equations which may be recast in a single matrix equation of the following form

$$\begin{bmatrix} \mathbf{K}_1 + \mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1 + \mathbf{K}_4 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_c \\ \mathbf{U}_s \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c,z} \\ \mathbf{U}_{s,z} \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_6 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c,zz} \\ \mathbf{U}_{s,zz} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_c \\ \mathbf{F}_s \end{Bmatrix}. \tag{6}$$

Recall that $\mathbf{K}_1$, $\mathbf{K}_4$, $\mathbf{K}_5$ and $\mathbf{K}_6$ are symmetric, while $\mathbf{K}_2$ and $\mathbf{K}_3$ are antisymmetric so that the first and third matrices in Eq. (6) are symmetric and the middle one antisymmetric. Substitution of $\mathbf{u}$ into the strain-displacement equations gives

$$\boldsymbol{\epsilon}(\theta, z) = [\boldsymbol{\epsilon}_c(z), \boldsymbol{\epsilon}_s(z)] \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{7}$$

with

$$\boldsymbol{\epsilon}_c(z) = \mathbf{b}_r\mathbf{u}_c + \mathbf{b}_\theta\mathbf{u}_s + \mathbf{b}_z\mathbf{u}_{c,z}, \qquad \boldsymbol{\epsilon}_s(z) = \mathbf{b}_r\mathbf{u}_s - \mathbf{b}_\theta\mathbf{u}_c + \mathbf{b}_z\mathbf{u}_{s,z}, \tag{8}$$

where strain-transformation matrices $\mathbf{b}_i$'s may be found in Zhuang et al. (1999). By means of the linear anisotropic constitutive equation, $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}$, the sinusoidal stress components are

$$\boldsymbol{\sigma}_c(z) = \mathbf{C}[\mathbf{b}_r\mathbf{u}_c + \mathbf{b}_\theta\mathbf{u}_s + \mathbf{b}_z\mathbf{u}_{c,z}], \qquad \boldsymbol{\sigma}_s(z) = \mathbf{C}[\mathbf{b}_r\mathbf{u}_s - \mathbf{b}_\theta\mathbf{u}_c + \mathbf{b}_z\mathbf{u}_{s,z}]. \tag{9}$$

The solution procedure rests on expressing $\mathbf{F}_c(z)$ and $\mathbf{F}_s(z)$ in a power series of z.

$$\begin{Bmatrix} \mathbf{F}_c(z) \\ \mathbf{F}_s(z) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}_{c1} \\ \mathbf{F}_{s1} \end{Bmatrix} + z\begin{Bmatrix} \mathbf{F}_{c2} \\ \mathbf{F}_{s2} \end{Bmatrix} + z^2\begin{Bmatrix} \mathbf{F}_{c3} \\ \mathbf{F}_{s3} \end{Bmatrix} + \cdots + z^n\begin{Bmatrix} \mathbf{F}_{cn} \\ \mathbf{F}_{sn} \end{Bmatrix}. \tag{10}$$

The response $\mathbf{U}$ is taken in a comparable series as

$$\begin{Bmatrix} \mathbf{U}_c(z) \\ \mathbf{U}_s(z) \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_{c0}(z) \\ \mathbf{U}_{s0}(z) \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_{c1}(z) \\ \mathbf{U}_{s1}(z) \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_{c2}(z) \\ \mathbf{U}_{s2}(z) \end{Bmatrix} + \cdots + \begin{Bmatrix} \mathbf{U}_{cn}(z) \\ \mathbf{U}_{sn}(z) \end{Bmatrix}. \tag{11}$$

Substitution of Eqs. (10) and (11) into Eq. (6) leads to an equation which can be grouped in a series of terms, each of which constitutes a problem in itself. All problems except for the first one are of the form

$$\begin{bmatrix} \mathbf{K}_1 + \mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1 + \mathbf{K}_4 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{ck} \\ \mathbf{U}_{sk} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{ck,z} \\ \mathbf{U}_{sk,z} \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_6 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{ck,zz} \\ \mathbf{U}_{sk,zz} \end{Bmatrix} = z^{k-1}\begin{Bmatrix} \mathbf{F}_{ck} \\ \mathbf{F}_{sk} \end{Bmatrix}. \tag{12}$$

The first equation governing $\mathbf{U}_{c0}$ and $\mathbf{U}_{s0}$ involve the homogeneous form of Eq. (12) since the first load term in Eq. (10) is identically zero. The total response requires the sequential analysis of these problems as data from the lower order terms are needed to construct the solutions of the higher order terms. The first problem relates to pure bending where the stress state is uniform along the cylinder's axis. For pure bending, there are no applied surface tractions. The second set involving $\mathbf{U}_{c1}$ and $\mathbf{U}_{s1}$ pertains to flexure by applied transverse forces on the tip end of the cylinder. The corresponding stress state is at most linear in z. The sinusoidal load terms $\mathbf{F}_{c1}$ and $\mathbf{F}_{s1}$ are longitudinal shears that amount to resultant uniform moments and such loads can be accommodated by a linear stress state. The next set, $\mathbf{U}_{c2}$ and $\mathbf{U}_{s2}$, leads to a stress state that varies quadratically in the axial direction. The force components $\mathbf{F}_{c2}$, $\mathbf{F}_{s2}$ are tractions with linearly varying resultant bending moments. These loads are consistent with a quadratically varying stress state. Higher displacement terms pertain to commensurately higher resultant loads and stress states. Herein, the first three cases are considered and they are designated as Problems I–III.

End conditions and global equilibrium equations involve pairs of orthogonal moments, $M_x(z)$ and $M_y(z)$ and transverse shears, $V_y(z)$ and $V_x(z)$, where x and y are two orthogonal axes in the cross-sectional plane of the cylinder. The moments at any station along the axis of the cylinder are obtained by integrating the axial stress σ_{zz} and their respective moment arms over the cross-section, i.e.,

$$\begin{aligned} M_x(z) &= -\int\int_A \sigma_{zz}\, y\,\mathrm{d}A = -\int_{r_\mathrm{i}}^{r_\mathrm{o}}\int_0^{2\pi} \sigma_{\mathrm{c}zz} r^2 \cos^2\theta\,\mathrm{d}r\,\mathrm{d}\theta, \\ M_y(z) &= -\int\int_A \sigma_{zz}\, x\,\mathrm{d}A = -\int_{r_\mathrm{i}}^{r_\mathrm{o}}\int_0^{2\pi} \sigma_{\mathrm{s}zz} r^2 \sin^2\theta\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned} \tag{13}$$

The shear forces are obtained by integrating the shear stresses σ_{zr} and $\sigma_{z\theta}$ over the cross-section, i.e.,

$$\begin{aligned} V_y(z) &= \int\int_A (\sigma_{zr}\cos\theta - \sigma_{z\theta}\sin\theta)\,\mathrm{d}A = \int_{r_\mathrm{i}}^{r_\mathrm{o}}\int_0^{2\pi} (\sigma_{\mathrm{c}zr}\cos^2\theta - \sigma_{\mathrm{s}z\theta}\sin^2\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta, \\ V_x(z) &= -\int\int_A (\sigma_{zr}\sin\theta + \sigma_{z\theta}\cos\theta)\,\mathrm{d}A = -\int_{r_\mathrm{i}}^{r_\mathrm{o}}\int_0^{2\pi} (\sigma_{\mathrm{s}zr}\sin^2\theta + \sigma_{\mathrm{c}z\theta}\cos^2\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned} \tag{14}$$

3. Rigid body displacement

Four rigid body motions, two lateral translations (u_0, v_0) and two rotations (ω_1, ω_2) about two orthogonal axes in the cross-sectional plane, satisfy the homogeneous form of Eq. (6) identically and their counterparts on the element level yield zero strains when substituted in Eq. (7). These identities are used in the solution procedure. Their sinusoidal components have the form

$$\begin{Bmatrix} \mathbf{U}_{RB\mathrm{c}} \\ \mathbf{U}_{RB\mathrm{s}} \end{Bmatrix} = u_0 \begin{Bmatrix} \mathbf{R}_1 \\ -\mathbf{R}_2 \end{Bmatrix} + v_0 \begin{Bmatrix} \mathbf{R}_2 \\ \mathbf{R}_1 \end{Bmatrix} + \omega_1 \left[z \begin{Bmatrix} \mathbf{R}_2 \\ \mathbf{R}_1 \end{Bmatrix} - \begin{Bmatrix} \mathbf{0} \\ \mathbf{R}_5 \end{Bmatrix} \right] + \omega_2 \left[z \begin{Bmatrix} \mathbf{R}_1 \\ -\mathbf{R}_2 \end{Bmatrix} - \begin{Bmatrix} \mathbf{R}_5 \\ \mathbf{0} \end{Bmatrix} \right], \tag{15}$$

where $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_5$ are given by

$$\mathbf{R}_1 = \begin{Bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \qquad \mathbf{R}_2 = \begin{Bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{Bmatrix}, \qquad \mathbf{R}_5 = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{r} \end{Bmatrix}. \tag{16}$$

The arrays $\mathbf{I}$ and $\mathbf{r}$ in Eq. (16) are of size $(N \times 1)$ containing N unit entries and N radial coordinates of the finite element model, respectively. Substituting the four rigid body displacements (15) into Eq. (11) leads to

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{Bmatrix} \mathbf{R}_1 \\ -\mathbf{R}_2 \end{Bmatrix} = 0, \qquad \begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{Bmatrix} \mathbf{R}_2 \\ \mathbf{R}_1 \end{Bmatrix} = 0,$$
$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{Bmatrix} \mathbf{0} \\ \mathbf{R}_5 \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix}\begin{Bmatrix} \mathbf{R}_2 \\ \mathbf{R}_1 \end{Bmatrix} = 0, \tag{17}$$
$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{Bmatrix} \mathbf{R}_5 \\ \mathbf{0} \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix}\begin{Bmatrix} \mathbf{R}_1 \\ -\mathbf{R}_2 \end{Bmatrix} = 0$$

and the strain-transformation equations are

$$[\mathbf{b}_r, \mathbf{b}_\theta]\begin{Bmatrix} \mathbf{r}_1 \\ -\mathbf{r}_2 \end{Bmatrix} = 0, \qquad [\mathbf{b}_r, \mathbf{b}_\theta]\begin{Bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \end{Bmatrix} = 0, \qquad [\mathbf{b}_\theta, \mathbf{b}_z]\begin{Bmatrix} \mathbf{r}_5 \\ \mathbf{r}_2 \end{Bmatrix} = 0, \qquad [\mathbf{b}_r, \mathbf{b}_z]\begin{Bmatrix} \mathbf{r}_5 \\ -\mathbf{r}_1 \end{Bmatrix} = 0. \tag{18}$$

4. Problem I – pure bending

Pure bending of a cylinder is characterized by strain and stress fields that are uniform along the axis of the cylinder. The sinusoidal components $\mathbf{U}_{c0}$ and $\mathbf{U}_{s0}$ satisfy the governing equation

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{Bmatrix} \mathbf{U}_{c0} \\ \mathbf{U}_{s0} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix}\begin{Bmatrix} \mathbf{U}_{c0,z} \\ \mathbf{U}_{s0,z} \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_6 \end{bmatrix}\begin{Bmatrix} \mathbf{U}_{c0,zz} \\ \mathbf{U}_{s0,zz} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{19}$$

The most general displacement forms of $\mathbf{U}_{c0}$ and $\mathbf{U}_{s0}$ for pure bending are

$$\begin{aligned}
u_{c0}(r,z) &= a_{I4}\left[-\frac{z^2}{2}+\psi_{cI4u}(r)\right] + a_{I5}\psi_{cI5u}(r),\\
u_{s0}(r,z) &= a_{I4}\psi_{sI4u}(r) + a_{I5}\left[-\frac{z^2}{2}+\psi_{sI5u}(r)\right],\\
v_{c0}(r,z) &= a_{I4}\psi_{cI5v}(r) + a_{I5}\left[-\frac{z^2}{2}+\psi_{cI5v}(r)\right],\\
v_{s0}(r,z) &= a_{I4}\left[-\frac{z^2}{2}+\psi_{sI5v}(r)\right] + a_{I5}\psi_{sI5v}(r),\\
w_{c0}(r,z) &= a_{I4}[zr+\psi_{cI4w}(r)] + a_{I5}\psi_{cI5w}(r),\\
w_{s0}(r,z) &= a_{I4}\psi_{sI4w}(r) + a_{I5}[zr+\psi_{sI5w}(r)],
\end{aligned} \tag{20a}$$

or in matrix notation as

$$\begin{Bmatrix} \mathbf{U}_{c0}(z) \\ \mathbf{U}_{s0}(z) \end{Bmatrix} = a_{I4}\begin{Bmatrix} \mathbf{U}_{cI4} \\ \mathbf{U}_{sI4} \end{Bmatrix} + a_{I5}\begin{Bmatrix} \mathbf{U}_{cI5} \\ \mathbf{U}_{sI5} \end{Bmatrix}, \tag{20b}$$

where

$$\begin{aligned}
\begin{Bmatrix} \mathbf{U}_{cI4} \\ \mathbf{U}_{sI4} \end{Bmatrix} &= -\frac{z^2}{2}\begin{Bmatrix} \mathbf{R}_1 \\ -\mathbf{R}_2 \end{Bmatrix} + z\begin{Bmatrix} \mathbf{R}_5 \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Psi}_{cI4} \\ \boldsymbol{\Psi}_{sI4} \end{Bmatrix},\\
\begin{Bmatrix} \mathbf{U}_{cI5} \\ \mathbf{U}_{sI5} \end{Bmatrix} &= -\frac{z^2}{2}\begin{Bmatrix} \mathbf{R}_2 \\ \mathbf{R}_1 \end{Bmatrix} + z\begin{Bmatrix} \mathbf{0} \\ \mathbf{R}_5 \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Psi}_{cI5} \\ \boldsymbol{\Psi}_{sI5} \end{Bmatrix}.
\end{aligned} \tag{21}$$

The $z^2/2$ and zr terms in Eq. (20a) constitute the primal field that express the Bernoulli–Euler kinematic hypothesis in cylindrical coordinates, and ψ_{cIij} and ψ_{sIij} define the warpages of the cross-section. Note that

Eqs. (20a) and (20b) may be obtained by integrating the rigid body rotations in Eq. (15) once with respect to z, a methodology due to Ieşan (1987).

Substituting Eqs. (20a) and (20b) into Eq. (19) gives

$$\begin{aligned}
&-a_{I4}\frac{z^2}{2}\left(\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\mathbf{R}_2\\ \mathbf{R}_1\end{Bmatrix}\right)-a_{I5}\frac{z^2}{2}\left(\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\mathbf{R}_1\\ -\mathbf{R}_2\end{Bmatrix}\right)\\
&-a_{I4}z\left(\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\mathbf{0}\\ \mathbf{R}_5\end{Bmatrix}+\begin{bmatrix}\mathbf{K}_3 & -\mathbf{K}_5\\ \mathbf{K}_5 & \mathbf{K}_3\end{bmatrix}\begin{Bmatrix}\mathbf{R}_1\\ -\mathbf{R}_2\end{Bmatrix}\right)\\
&+a_{I5}z\left(\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\mathbf{R}_5\\ \mathbf{0}\end{Bmatrix}-\begin{bmatrix}\mathbf{K}_3 & -\mathbf{K}_5\\ \mathbf{K}_5 & \mathbf{K}_3\end{bmatrix}\begin{Bmatrix}\mathbf{R}_2\\ \mathbf{R}_1\end{Bmatrix}\right)\\
&+a_{I4}\left(\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\boldsymbol{\Psi}_{cI4}\\ \boldsymbol{\Psi}_{sI4}\end{Bmatrix}+\begin{Bmatrix}\mathbf{K}_3\mathbf{R}_5+\mathbf{K}_6\mathbf{R}_1\\ \mathbf{K}_5\mathbf{R}_5-\mathbf{K}_6\mathbf{R}_2\end{Bmatrix}\right)\\
&+a_{I5}\left(\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\boldsymbol{\Psi}_{cI5}\\ \boldsymbol{\Psi}_{sI5}\end{Bmatrix}+\begin{Bmatrix}-\mathbf{K}_5\mathbf{R}_5+\mathbf{K}_6\mathbf{R}_2\\ \mathbf{K}_3\mathbf{R}_5+\mathbf{K}_6\mathbf{R}_1\end{Bmatrix}\right)=0.
\end{aligned}\tag{22}$$

Each term of Eq. (22) enclosed by parentheses must vanish. The terms multiplied by z^2 and z are satisfied by the rigid body identities (17). The remaining terms provide means for determining the warpages.

$$\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\boldsymbol{\Psi}_{cI4}\\ \boldsymbol{\Psi}_{sI4}\end{Bmatrix}=-\begin{Bmatrix}\mathbf{K}_3\mathbf{R}_5+\mathbf{K}_6\mathbf{R}_1\\ \mathbf{K}_5\mathbf{R}_5-\mathbf{K}_6\mathbf{R}_2\end{Bmatrix},\tag{23a}$$

$$\begin{bmatrix}\mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4\end{bmatrix}\begin{Bmatrix}\boldsymbol{\Psi}_{cI5}\\ \boldsymbol{\Psi}_{sI5}\end{Bmatrix}=-\begin{Bmatrix}-\mathbf{K}_5\mathbf{R}_5+\mathbf{K}_6\mathbf{R}_2\\ \mathbf{K}_3\mathbf{R}_5+\mathbf{K}_6\mathbf{R}_1\end{Bmatrix}.\tag{23b}$$

These warpage functions are driven by stiffness matrix terms $\mathbf{K}_3$, $\mathbf{K}_5$, and $\mathbf{K}_6$ operating on the components of the primal field. A comparison of this pair of equations reveals that

$$\boldsymbol{\Psi}_{cI4}=\boldsymbol{\Psi}_{sI5},\qquad \boldsymbol{\Psi}_{sI4}=-\boldsymbol{\Psi}_{cI5}.\tag{24}$$

This equivalence of the warpages about two orthogonal axes is manifestation of material axial symmetry, as the terms differ only by a $\pi/2$ rotation by the z-axis. Therefore, only one equation in the set of Eqs. (23a) and (23b) needs to be solved to establish the warpages. This pattern of identical solutions about the two orthogonal axes is repeated is subsequent problems, so that only one series need to be solved.

The stiffness matrix in Eq. (23a) or Eq. (23b) is singular by two degrees of freedom due to two rigid body displacements. These two motions must be suppressed in order for a unique inverse. Imposing this restraint does not affect the relative displacements of deformation, as a rigid body displacement can always be appended to meet the clamped end boundary conditions. The choice of these degrees of freedom relates to the locations of the applied normal and circumferential surface tractions in subsequent problems.

Substituting the element level of displacement field (20a) and (20b) into the stress transformation equations (9) yields the following sinusoidal components of the stress transformation equations.

$$\boldsymbol{\sigma}_{c0}=\mathbf{C}[\mathbf{b}_r\boldsymbol{\psi}_{cI4}+\mathbf{b}_\theta\boldsymbol{\psi}_{sI4}+\mathbf{b}_z\mathbf{r}_5],\qquad \boldsymbol{\sigma}_{s0}=\mathbf{C}[\mathbf{b}_r\boldsymbol{\psi}_{sI4}-\mathbf{b}_\theta\boldsymbol{\psi}_{cI4}],\tag{25}$$

where rigid body identities (18) were invoked. Observe that stress field (25) is independent of z, verifying the pure bending state.

The coefficients a_{I4} and a_{I5} are found from the applied moments M_x and M_y. Substituting σ_{zz} from Eq. (25) into Eq. (13) gives

$$\begin{Bmatrix}M_x\\ M_y\end{Bmatrix}=-\begin{bmatrix}\kappa_{I44} & 0\\ 0 & \kappa_{I44}\end{bmatrix}\begin{Bmatrix}a_{I4}\\ a_{I5}\end{Bmatrix}\tag{26}$$

with κ_{I44} as the flexural rigidity of the circular cross-section. The absence of the off-diagonal term κ_{I45} is due to orthogonality of $\sin\theta$ and $\cos\theta$ with σ_{zz} in the integrands of Eq. (13) and axial symmetry of the circular cross-section. Solution of Eq. (26) gives

$$a_{I4} = -\frac{M_x}{\kappa_{I44}}, \qquad a_{I5} = -\frac{M_y}{\kappa_{I44}}. \tag{27}$$

5. Problem II – generalized flexure

Generalized flexure herein refers to strain and stress fields that are at most linear in the axial direction. The governing equation for this case is

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{Bmatrix} \mathbf{U}_{c1} \\ \mathbf{U}_{s1} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix}\begin{Bmatrix} \mathbf{U}_{c1,z} \\ \mathbf{U}_{s1,z} \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_6 \end{bmatrix}\begin{Bmatrix} \mathbf{U}_{c1,zz} \\ \mathbf{U}_{s1,zz} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{c1} \\ \mathbf{F}_{s1} \end{Bmatrix}. \tag{28}$$

In generalized flexure, there are two loading conditions: (1) applied transverse shear forces P_x and P_y at $z=0$, and (2) uniform axial longitudinal surface shear tractions (p_{czi}, p_{czo}) and (p_{szi}, p_{szo}) of sinusoidal circumferential variations on the inner and outer surfaces, (r_i, r_o) as shown in Fig. 1b. These tractions are contained in $\mathbf{F}_{c1}$ and $\mathbf{F}_{s1}$ and they possess resultant moments per unit length (m_x, m_y) of

$$\begin{aligned} m_x &= \int_0^{2\pi} [p_{0zco} r_o^2 - p_{czi} r_i^2] \cos^2\theta \, d\theta = \pi[p_{czo} r_o^2 - p_{czi} r_i^2], \\ m_y &= \int_0^{2\pi} [p_{0zso} r_o^2 - p_{szi} r_i^2] \sin^2\theta \, d\theta = \pi[p_{szo} r_o^2 - p_{szi} r_i^2]. \end{aligned} \tag{29}$$

Assume that M_x and M_y at $z=0$ are zero, as end tractions giving an end moment can be treated in Problem I.

For generalized flexure, the most general form of the sinusoidal components of the displacement field in matrix notation is

$$\begin{Bmatrix} U_{c1}(z) \\ U_{s1}(z) \end{Bmatrix} = a_{II4}\begin{Bmatrix} \mathbf{U}_{cII4} \\ \mathbf{U}_{sII4} \end{Bmatrix} + b_{II4}\begin{Bmatrix} \mathbf{U}_{cI4} \\ \mathbf{U}_{sI4} \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_{cII4p} \\ \mathbf{U}_{sII4p} \end{Bmatrix} + a_{II5}\begin{Bmatrix} \mathbf{U}_{cII5} \\ \mathbf{U}_{sII5} \end{Bmatrix} + b_{II5}\begin{Bmatrix} \mathbf{U}_{cI5} \\ \mathbf{U}_{sI5} \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_{cII5p} \\ \mathbf{U}_{sII5p} \end{Bmatrix}, \tag{30}$$

where $(\mathbf{U}_{cI4}, \mathbf{U}_{sI4}, \mathbf{U}_{cI5}, \mathbf{U}_{sI5})$ are components of Problem I. Deformation coefficients $(a_{II4}, a_{II5}, b_{II4}, b_{II5})$ are new coefficients, $(\mathbf{U}_{cII4p}, \mathbf{U}_{sII4p})$ and $(\mathbf{U}_{cII5p}, \mathbf{U}_{sII5p})$ are particular solutions for the applied longitudinal surface shears, and $(\mathbf{U}_{cII4}, \mathbf{U}_{sII4}, \mathbf{U}_{cII5}, \mathbf{U}_{sII5})$ are new components given by

$$\begin{aligned} \begin{Bmatrix} \mathbf{U}_{cII4} \\ \mathbf{U}_{sII4} \end{Bmatrix} &= -\frac{z^3}{6}\begin{Bmatrix} \boldsymbol{R}_1 \\ -\boldsymbol{R}_2 \end{Bmatrix} + \frac{z^2}{2}\begin{Bmatrix} \boldsymbol{R}_5 \\ 0 \end{Bmatrix} + z\begin{Bmatrix} \boldsymbol{\Psi}_{cI4} \\ \boldsymbol{\Psi}_{sI4} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Psi}_{cII4} \\ \boldsymbol{\Psi}_{sII4} \end{Bmatrix}, \\ \begin{Bmatrix} \mathbf{U}_{cII5} \\ \mathbf{U}_{sII5} \end{Bmatrix} &= -\frac{z^3}{6}\begin{Bmatrix} \boldsymbol{R}_2 \\ \boldsymbol{R}_1 \end{Bmatrix} + \frac{z^2}{2}\begin{Bmatrix} 0 \\ \boldsymbol{R}_5 \end{Bmatrix} + z\begin{Bmatrix} \boldsymbol{\Psi}_{cI5} \\ \boldsymbol{\Psi}_{sI5} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Psi}_{cII5} \\ \boldsymbol{\Psi}_{sII5} \end{Bmatrix}. \end{aligned} \tag{31}$$

Warpages $(\boldsymbol{\Psi}_{cI4}, \boldsymbol{\Psi}_{sI4}, \boldsymbol{\Psi}_{cI5}, \boldsymbol{\Psi}_{sI5})$ are those of Problem I, but $(\boldsymbol{\Psi}_{cII4}, \boldsymbol{\Psi}_{sII4}, \boldsymbol{\Psi}_{cII5}, \boldsymbol{\Psi}_{sII5})$ are new. Eq. (31) may be obtained by integrating Eq. (21) once with respect to z, a procedure due to Ieşan (1987).

For clarity of discussion, consider first the flexure of the cylinder by a shear force applied at its tip end without applied longitudinal surface tractions. Thus, the particular solutions in Eq. (30) are not involved. Substituting this truncated form of displacement field (30) into homogeneous Eq. (28) gives a host of terms, many of which are satisfied by the rigid body displacement identities and equations of Problem I. For brevity sake, these expressions are not shown here. The new terms provide equations for determining the new warpage functions. The set of equations for $\boldsymbol{\Psi}_{cII4}$ and $\boldsymbol{\Psi}_{sII4}$ is

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_{cII4} \\ \boldsymbol{\Psi}_{sII4} \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_3\boldsymbol{\Psi}_{cI4} - \mathbf{K}_5\boldsymbol{\Psi}_{sI4} - \mathbf{K}_6\mathbf{R}_5 \\ \mathbf{K}_5\boldsymbol{\Psi}_{cI4} + \mathbf{K}_3\boldsymbol{\Psi}_{sI4} \end{bmatrix}. \tag{32}$$

There is an analogous set of equations for $\boldsymbol{\Psi}_{cII5}$ and $\boldsymbol{\Psi}_{sII5}$ which is not shown. Its solution shows that $\boldsymbol{\Psi}_{sII5} = \boldsymbol{\Psi}_{cII4}$ and $\boldsymbol{\Psi}_{cII5} = -\boldsymbol{\Psi}_{sII4}$, a property due to material axial symmetry that was mentioned previously.

The sinusoidal components of the stress field from solution for the displacements and stress displacement relation (8) are

$$\begin{aligned} \boldsymbol{\sigma}_{c1} &= a_{II4}[z\boldsymbol{\sigma}_{c0} + \boldsymbol{\sigma}_{c1}] - a_{II5}[z\boldsymbol{\sigma}_{s0} + \boldsymbol{\sigma}_{s1}] + b_{II4}\boldsymbol{\sigma}_{c0} - b_{II5}\boldsymbol{\sigma}_{s0}, \\ \boldsymbol{\sigma}_{s1} &= a_{II4}[z\boldsymbol{\sigma}_{s0} + \boldsymbol{\sigma}_{s1}] + a_{II5}[z\boldsymbol{\sigma}_{c0} + \boldsymbol{\sigma}_{c1}] + b_{II4}\boldsymbol{\sigma}_{s0} + b_{II5}\boldsymbol{\sigma}_{c0}, \end{aligned} \tag{33}$$

where $(\boldsymbol{\sigma}_{c0}, \boldsymbol{\sigma}_{s0})$ were given by Eq. (25) and

$$\boldsymbol{\sigma}_{c1} = \mathbf{C}[\boldsymbol{b}_r\boldsymbol{\psi}_{cII4} + \boldsymbol{b}_\theta\boldsymbol{\psi}_{sII4} + \boldsymbol{b}_z\boldsymbol{\psi}_{cI4}], \qquad \boldsymbol{\sigma}_{s1} = \mathbf{C}[\boldsymbol{b}_r\boldsymbol{\psi}_{sII4} - \boldsymbol{b}_\theta\boldsymbol{\psi}_{cII4} + \boldsymbol{b}_z\boldsymbol{\psi}_{sI4}]. \tag{34}$$

Integrating σ_{zz} of Eq. (33) according to Eq. (13) yields M_x and M_y as

$$\begin{Bmatrix} M_x(z) \\ M_y(z) \end{Bmatrix} = -\left(z\begin{bmatrix} \kappa_{I44} & 0 \\ 0 & \kappa_{I44} \end{bmatrix} + \begin{bmatrix} 0 & \kappa_{II45} \\ -\kappa_{II45} & 0 \end{bmatrix} \right) \begin{Bmatrix} a_{II4} \\ a_{II5} \end{Bmatrix} - \begin{bmatrix} \kappa_{I44} & 0 \\ 0 & \kappa_{I44} \end{bmatrix} \begin{Bmatrix} b_{II4} \\ b_{II5} \end{Bmatrix}. \tag{35}$$

Observe that $\boldsymbol{\kappa}_{\mathbf{II}}$ is antisymmetric.

Coefficients a_{II4} and a_{II5} are determined from global equilibrium that relates shear forces (P_x, P_y) to the rate of change of moments (M_x, M_y).

$$-\frac{\partial M_x}{\partial z} = P_y, \qquad -\frac{\partial M_y}{\partial z} = P_x. \tag{36}$$

Invoking this condition by differentiating Eq. (35) furnishes a_{II4} and a_{II5} as

$$a_{II4} = \frac{P_y}{\kappa_{I44}}, \qquad a_{II5} = \frac{P_x}{\kappa_{I44}}. \tag{37}$$

Substituting a_{II4} and a_{II5} into Eq. (35) and invoking the zero moment initial condition at $z = 0$ give b_{II4} and b_{II5} as

$$b_{II4} = -\frac{\kappa_{II45}}{\kappa_{I44}} a_{II5}, \qquad b_{II5} = \frac{\kappa_{II45}}{\kappa_{I44}} a_{II4}. \tag{38}$$

For a homogeneous, isotropic cylinder, κ_{II45} is identically zero so that $b_{II4} = b_{II5} = 0$ and the only stress components uniform in z are shears $\sigma_{\theta z}$ and σ_{rz}. In contrast, there is a uniform axial stress σ_{zz} that accompanies the shear stresses in an anisotropic cylinder. The integral of this σ_{zz} distribution over the cross-section leads to a pure bending moment with a vectorial direction parallel to that of the shear force. Global equilibrium is nevertheless maintained as this moment is negated by an equal but opposite pure bending moment through the presence of κ_{II45} that relates the a_{IIi}'s to the b_{IIi}'s by Eq. (38). The b_{IIi}'s terms provide for the equilibrating pure bending moment as they are connected to stresses $\boldsymbol{\sigma}_{c0}$ and $\boldsymbol{\sigma}_{s0}$ in Eq. (33), which contain the essence of this moment. However, the superposition of these two σ_{zz} distributions leaves a self-equilibrated stress state rather than a traction-free surface. The self-equilibrated stress state decays with distance into the interior of the cylinder according to Saint-Venant's Principle, and means for quantifying the manner of this decay are mentioned in the concluding remarks. By the relaxed formulation, end conditions in an anisotropic cylinder can only be achieved on an integral basis. A numerical example on the flexure of an anisotropic cylinder will illustrate this phenomenon of self-equilibrated stresses from two pure bending moments more clearly.

Now consider applied longitudinal surface shear tractions on the cylinder which are contained in load vectors $(\mathbf{F}_{c0}, \mathbf{F}_{s0})$ in Eq. (28). They possess resultant uniform moments per unit length m_x and m_y given by

Eq. (29). These resultant moments cause changes in M_x and M_y, so that the analogue to global equilibrium equation (36) for this loading condition is

$$-\frac{\partial M_x}{\partial z} = m_x, \qquad -\frac{\partial M_y}{\partial z} = m_y \tag{39}$$

with m_x and m_y in place of the transverse forces. Using particular solutions $\mathbf{U}_{cII4p}$ and $\mathbf{U}_{sII4p}$ in Eq. (28) gives

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{cII4p} \\ \mathbf{U}_{sII4p} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{c0} \\ \mathbf{F}_{s0} \end{bmatrix}. \tag{40}$$

Similarly particular solutions $\mathbf{U}_{cII5p}$ and $\mathbf{U}_{sII5p}$ depend on a set of equations analogous to (40), which shows that $\mathbf{U}_{sII5p} = \mathbf{U}_{cII4p}$ and $\mathbf{U}_{cII5p} = -\mathbf{U}_{sII4p}$. Thus, manifestation of axial symmetry is again seen.

The sinusoidal components of the stress field of the particular solutions are

$$\boldsymbol{\sigma}_{cp1} = \mathbf{C}\big[\boldsymbol{b}_r\mathbf{u}_{cII4p} + \boldsymbol{b}_\theta\mathbf{u}_{sII4p}\big], \qquad \boldsymbol{\sigma}_{sp1} = \boldsymbol{C}\big[\boldsymbol{b}_r\mathbf{u}_{sII4p} - \boldsymbol{b}_\theta\mathbf{u}_{cII4p} bigr\big] \tag{41}$$

Integrating $\sigma_{z\theta}$ and σ_{rz} of Eq. (41) over the cross-section according to formula (14) gives a shear resultant V_y that is equal in magnitude to moment per unit length m_x, thus satisfying global equilibrium (39). However, a transverse shear force V_y cannot exist for loading from longitudinal shear only. Negating this force requires the superposition of an equal but opposite flexure force of the generalized flexure solution. Even though this superposition of these transverse force gives a zero net, the net shear stresses do not vanish on a point-wise basis but comprise a self-equilibrated set that decays into the interior according to Saint-Venant's principle.

6. Problem III – linearly varying flexure

This problem is characterized by stress and strain fields that vary quadratically with z and is governed by

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c2} \\ \mathbf{U}_{s2} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c2,z} \\ \mathbf{U}_{s2,z} \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_6 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c2,zz} \\ \mathbf{U}_{s2,zz} \end{Bmatrix} = z\begin{Bmatrix} \mathbf{F}_{c2} \\ \mathbf{F}_{s2} \end{Bmatrix}. \tag{42}$$

Three traction components of sinusoidal circumferential variation may occur on the inner and outer surfaces: (1) linearly varying longitudinal shears zp'_{czi} and zp'_{szo} (Fig. 1a), (2) normal pressures p_{cri} and p_{sro} (Fig. 1b), and (3) circumferential shears $p_{c\theta i}$ and $p_{s\theta o}$ (Fig. 1b). The linearly varying longitudinal shears lack a resultant axial force, but possess linearly varying moment resultants zm'_x and zm'_y about both axes where m'_x and m'_y are given by

$$\begin{aligned} m'_x &= \int_0^{2\pi} \big[p'_{czo}r_o^2 - p'_{czi}r_i^2\big]\cos^2\theta\,d\theta = \pi\big[p'_{czo}r_o^2 - p'_{czi}r_i^2\big], \\ m'_y &= \int_0^{2\pi} \big[p'_{szo}r_o^2 - p'_{szi}r_i^2\big]\sin^2\theta\,d\theta = \pi\big[p'_{szo}r_o^2 - p'_{szi}r_i^2\big]. \end{aligned} \tag{43}$$

The resultants due to normal pressures p_{cri} and p_{sro} and circumferential shears $p_{c\theta i}$ and $p_{s\theta o}$ are uniform transverse loads p_x and p_y of magnitudes

$$\begin{aligned} p_x &= \int_0^{2\pi} \big[r_o\big[p_{cro}\cos^2\theta - p_{s\theta o}\sin^2\theta\big] - r_i\big[p_{cri}\cos^2\theta - p_{s\theta i}\sin^2\theta\big]\big]\,d\theta \\ &= \pi[r_o[p_{cro} - p_{s\theta o}] - r_i[p_{cri} - p_{s\theta i}]], \\ p_y &= \int_0^{2\pi} \big[r_o\big[p_{sro}\sin^2\theta - p_{c\theta o}\cos^2\theta\big] - r_i\big[p_{srs}\sin^2\theta - p_{c\theta i}\cos^2\theta\big]\big]\,d\theta \\ &= \pi[r_o[p_{sro} - p_{c\theta o}] - r_i[p_{sri} - p_{c\theta i}]]. \end{aligned} \tag{44}$$

For Problem III, zero moments and transverse forces at $z=0$ are taken, i.e., $M_x(0)=M_y(0)=0$ and $P_x(0)=P_y(0)=0$. Also, uniform longitudinal shears are not considered. All of these loads can be treated by superposing the results of Problems I and II.

The sinusoidal displacement components for Problem III have the form

$$\begin{Bmatrix} U_{c2}(z) \\ U_{s2}(z) \end{Bmatrix} = a_{III4}\begin{Bmatrix} \mathbf{U}_{cIII4} \\ \mathbf{U}_{sIII4} \end{Bmatrix} + b_{III4}\begin{Bmatrix} \mathbf{U}_{cII4} \\ \mathbf{U}_{sII4} \end{Bmatrix} + c_{III4}\begin{Bmatrix} \mathbf{U}_{cI4} \\ \mathbf{U}_{sI4} \end{Bmatrix} + z\begin{Bmatrix} \mathbf{U}_{cIII4p1} \\ \mathbf{U}_{sIII4p1} \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_{cIII4p2} \\ \mathbf{U}_{sIII4p2} \end{Bmatrix}$$
$$+ a_{III5}\begin{Bmatrix} \mathbf{U}_{cIII5} \\ \mathbf{U}_{sIII5} \end{Bmatrix} + b_{III5}\begin{Bmatrix} \mathbf{U}_{cII5} \\ \mathbf{U}_{sII5} \end{Bmatrix} + c_{III5}\begin{Bmatrix} \mathbf{U}_{cI5} \\ \mathbf{U}_{sI5} \end{Bmatrix} + z\begin{Bmatrix} \mathbf{U}_{cIII5p1} \\ \mathbf{U}_{sIII5p1} \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_{cIII5p2} \\ \mathbf{U}_{sIII5p2} \end{Bmatrix}, \tag{45}$$

where coefficients a_{IIIi}'s, b_{IIIi}'s, and c_{IIIi}'s are new. All displacement terms in Eq. (45) have been defined previously except for the terms $\mathbf{U}_{cIIIi}$'s and $\mathbf{U}_{sIIIi}$'s and particular solutions ($\mathbf{U}_{cIIIipj}$'s and $\mathbf{U}_{sIIIipj}$'s). The new terms $\mathbf{U}_{cIII4}$, $\mathbf{U}_{sIII4}$, $\mathbf{U}_{cIII5}$, and $\mathbf{U}_{sIII5}$ are given by

$$\begin{aligned}\begin{Bmatrix} \mathbf{U}_{cIII4} \\ \mathbf{U}_{sIII4} \end{Bmatrix} &= -\frac{z^4}{24}\begin{Bmatrix} \boldsymbol{R}_1 \\ -\boldsymbol{R}_2 \end{Bmatrix} - \frac{z^3}{6}\begin{Bmatrix} \boldsymbol{R}_5 \\ 0 \end{Bmatrix} + \frac{z^2}{2}\begin{Bmatrix} \boldsymbol{\Psi}_{cI4} \\ \boldsymbol{\Psi}_{sI4} \end{Bmatrix} + z\begin{Bmatrix} \boldsymbol{\Psi}_{cII4} \\ \boldsymbol{\Psi}_{sII4} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Psi}_{cIII4} \\ \boldsymbol{\Psi}_{sIII4} \end{Bmatrix}, \\ \begin{Bmatrix} \mathbf{U}_{cIII5} \\ \mathbf{U}_{sIII5} \end{Bmatrix} &= -\frac{z^4}{24}\begin{Bmatrix} \boldsymbol{R}_2 \\ \boldsymbol{R}_1 \end{Bmatrix} + \frac{z^3}{5}\begin{Bmatrix} 0 \\ \boldsymbol{R}_5 \end{Bmatrix} + \frac{z^2}{2}\begin{Bmatrix} \boldsymbol{\Psi}_{cI5} \\ \boldsymbol{\Psi}_{sI5} \end{Bmatrix} + z\begin{Bmatrix} \boldsymbol{\Psi}_{cII5} \\ \boldsymbol{\Psi}_{sII5} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Psi}_{cIII5} \\ \boldsymbol{\Psi}_{sIII5} \end{Bmatrix}\end{aligned} \tag{46}$$

with $\boldsymbol{\Psi}_{cIIIi}$'s and $\boldsymbol{\Psi}_{sIIIi}$'s as new warpage functions. Displacement field (46) can be obtained by integrating Eq. (31) once with respect to z, i.e., Ieşan's methodology (1987).

First, consider transverse loading by surface normal and shear tractions. The homogeneous form of Eq. (42) applies and the particular solutions in Eq. (45) are not involved. Substituting the truncated form of Eq. (45) into homogeneous equation (42) leads to a group of terms, with many of them satisfied by relations established in Problems I and II. The new expression associated with coefficient a_{III4} enables $\boldsymbol{\Psi}_{cIII4}$ and $\boldsymbol{\Psi}_{sIII4}$ to be determined.

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix}\begin{bmatrix} \boldsymbol{\Psi}_{cIII4} \\ \boldsymbol{\Psi}_{sIII4} \end{bmatrix} = -\begin{bmatrix} \mathbf{K}_3\boldsymbol{\Psi}_{cII4} - \mathbf{K}_5\boldsymbol{\Psi}_{sII4} - \mathbf{K}_6\boldsymbol{\Psi}_{cI4} \\ \mathbf{K}_5\boldsymbol{\Psi}_{cII4} + \mathbf{K}_3\boldsymbol{\Psi}_{sII4} - \mathbf{K}_6\boldsymbol{\Psi}_{sI4} \end{bmatrix}. \tag{47}$$

A similar consideration of the expression associated with a_{III5} shows that $\boldsymbol{\Psi}_{sIII5}=\boldsymbol{\Psi}_{cIII4}$ and $\boldsymbol{\Psi}_{cIII5}=-\boldsymbol{\Psi}_{sIII4}$.

The location of the normal and surface shear tractions depends on the two restraints chosen to render the stiffness matrix of Eq. (47) non-singular (recall that this matrix is singular by two degrees of freedom). For example, suppressing the radial and circumferential displacements on the outer surface yields warpages $\boldsymbol{\Psi}_{cIII4}$ and $\boldsymbol{\Psi}_{sIII4}$ that define the stress distribution for externally applied tractions p_{cro} and $p_{s\theta o}$. Similarly, inner surface restraints give warpages defining stress distributions for surface tractions p_{cri} and $p_{s\theta i}$ on the inner surface. A mixed combination of these displacements on inner and outer surfaces is also possible. This usage of the restraints was seen in the previous paper on axisymmetrical deformations where the warpages were said to have the roles of *applied distributed loads* and the suppressed degrees of freedom act as the *supports*.

The sinusoidal stress components in this case are

$$\begin{aligned}\begin{Bmatrix} \boldsymbol{\sigma}_{c1} \\ \boldsymbol{\sigma}_{s1} \end{Bmatrix} = {} & a_{III4}\left[\frac{z^2}{2}\begin{Bmatrix} \boldsymbol{\sigma}_{c0} \\ \boldsymbol{\sigma}_{s0} \end{Bmatrix} + z\begin{Bmatrix} \boldsymbol{\sigma}_{c1} \\ \boldsymbol{\sigma}_{s1} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\sigma}_{c2} \\ \boldsymbol{\sigma}_{s2} \end{Bmatrix}\right] + b_{III4}\left[z\begin{Bmatrix} \boldsymbol{\sigma}_{c0} \\ \boldsymbol{\sigma}_{s0} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\sigma}_{c1} \\ \boldsymbol{\sigma}_{s1} \end{Bmatrix}\right] + c_{III4}\begin{Bmatrix} \boldsymbol{\sigma}_{c0} \\ \boldsymbol{\sigma}_{s0} \end{Bmatrix} \\ & + a_{III5}\left[\frac{z^2}{2}\begin{Bmatrix} -\boldsymbol{\sigma}_{s0} \\ \boldsymbol{\sigma}_{c0} \end{Bmatrix} + z\begin{Bmatrix} -\boldsymbol{\sigma}_{s1} \\ \boldsymbol{\sigma}_{c1} \end{Bmatrix} + \begin{Bmatrix} -\boldsymbol{\sigma}_{s2} \\ \boldsymbol{\sigma}_{c2} \end{Bmatrix}\right] + b_{III5}\left[z\begin{Bmatrix} -\boldsymbol{\sigma}_{s0} \\ \boldsymbol{\sigma}_{c0} \end{Bmatrix} + \begin{Bmatrix} -\boldsymbol{\sigma}_{s1} \\ \boldsymbol{\sigma}_{c1} \end{Bmatrix}\right] \\ & + c_{III5}\begin{Bmatrix} -\boldsymbol{\sigma}_{s0} \\ \boldsymbol{\sigma}_{c0} \end{Bmatrix},\end{aligned} \tag{48}$$

where the new terms are

$$\begin{aligned}\boldsymbol{\sigma}_{c2} &= \mathbf{C}[\mathbf{b}_r\boldsymbol{\Psi}_{cIII4} + \mathbf{b}_\theta\boldsymbol{\Psi}_{sIII4} + \mathbf{b}_z\boldsymbol{\Psi}_{cII4}],\\ \boldsymbol{\sigma}_{s2} &= \mathbf{C}[\mathbf{b}_r\boldsymbol{\Psi}_{sIII4} - \mathbf{b}_\theta\boldsymbol{\Psi}_{cIII4} + \mathbf{b}_z\boldsymbol{\Psi}_{sII4}].\end{aligned} \tag{49}$$

Integrating σ_{zz} from Eq. (49) over the cross-section according to Eq. (13) gives M_x and M_y as

$$\begin{aligned}\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = &-\left(\frac{z^2}{2}\begin{bmatrix}\kappa_{I44} & 0\\ 0 & \kappa_{I44}\end{bmatrix} + z\begin{bmatrix}0 & \kappa_{II45}\\ -\kappa_{II45} & 0\end{bmatrix} + \begin{bmatrix}\kappa_{III44} & 0\\ 0 & \kappa_{III44}\end{bmatrix}\right)\begin{Bmatrix} a_{III4} \\ a_{III5}\end{Bmatrix} - \left(z\begin{bmatrix}\kappa_{I44} & 0\\ 0 & \kappa_{I44}\end{bmatrix}\right.\\ &\left.+\begin{bmatrix}0 & \kappa_{II45}\\ -\kappa_{II45} & 0\end{bmatrix}\right)\begin{Bmatrix} b_{III4} \\ b_{III5}\end{Bmatrix} - \begin{bmatrix}\kappa_{I44} & 0\\ 0 & \kappa_{I44}\end{bmatrix}\begin{Bmatrix} c_{III4} \\ c_{III5}\end{Bmatrix}.\end{aligned} \tag{50}$$

Coefficients a_{III4} and a_{III5} are determined from global equilibrium conditions that relate the second derivatives of the moments M_x and M_y to their respective resultant transverse forces in the two coordinate directions.

$$-\frac{\partial^2 M_x}{\partial z^2} = p_x, \qquad -\frac{\partial^2 M_y}{\partial z^2} = p_y, \tag{51}$$

where p_x and p_y are given by Eq. (43). Applying Eq. (51) to Eq. (50) gives a_{III4} and a_{III5} as

$$a_{III4} = \frac{p_y}{\kappa_{I44}}, \qquad a_{III5} = \frac{p_x}{\kappa_{I44}}. \tag{52}$$

The values of b_{III4} and b_{III5} come from the derivatives of M_x and M_y vanishing at $z = 0$ since there are no tip forces P_y and P_x.

$$\left.\begin{aligned} -\frac{\partial M_x}{\partial z}\Big|_{z=0} &= 0\\ -\frac{\partial M_y}{\partial z}\Big|_{z=0} &= 0\end{aligned}\right\} \quad \to \quad b_{III4} = -\frac{\kappa_{II45}}{\kappa_{I44}}a_{III5}, \qquad b_{III5} = \frac{\kappa_{II45}}{\kappa_{I44}}a_{III4}. \tag{53}$$

Lastly, invoking zero moments M_x and M_y at $z = 0$ in Eq. (50) gives c_{III4} and c_{III5} as

$$c_{III4} = -\frac{\kappa_{II45}}{\kappa_{I44}}b_{III5} - \frac{\kappa_{III44}}{\kappa_{I44}}a_{III4}, \qquad c_{III5} = \frac{\kappa_{II45}}{\kappa_{I44}}b_{III4} - \frac{\kappa_{III44}}{\kappa_{I44}}a_{III5}. \tag{54}$$

In this solution, the stresses σ_{c2} and σ_{s2} lead to resultant transverse shear forces and moments. These resultants cannot occur and they are negated by equal but opposite resultants through the b_{IIIi} and c_{IIIi} terms and the presence of κ_{II45} and κ_{III44} in Eqs. (53) and (54). These coefficients are associated with stresses that supply uniform moment and force resultants of Problems I and II for self-equilibration. While this combination satisfies equilibria on an integral basis, they do not leave a point-wise traction-free surface, but one which is self-equilibrated that decays with distance into the interior of the cylinder according to Saint-Venant's principle.

Now consider the linearly varying longitudinal surface shear traction zp'_{czi} that is contained $\mathbf{F}_{c2}$ of Eq. (42). The traction zp'_{szo} in $\mathbf{F}_{s1}$ of Eq. (42) may be omitted inasmuch as it causes a response identical to that of zp'_{czi} about the other axis. Using particular solution in Eq. (45) with subscript 4 in Eq. (42) gives the following two sets of equations that must be solved sequentially to establish the displacement field for longitudinal shear.

$$\begin{bmatrix}\mathbf{K}_1 + \mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1 + \mathbf{K}_4\end{bmatrix}\begin{bmatrix}\mathbf{U}_{cIII4p1}\\ \mathbf{U}_{sIII4p1}\end{bmatrix} = \begin{Bmatrix}\mathbf{F}_{c1}\\ \mathbf{0}\end{Bmatrix}, \tag{55}$$

$$\begin{bmatrix}\mathbf{K}_1 + \mathbf{K}_4 & \mathbf{K}_2\\ -\mathbf{K}_2 & \mathbf{K}_1 + \mathbf{K}_4\end{bmatrix}\begin{bmatrix}\mathbf{U}_{cIII4p2}\\ \mathbf{U}_{sIII4p2}\end{bmatrix} = -\begin{bmatrix}\mathbf{K}_3 & -\mathbf{K}_5\\ \mathbf{K}_5 & \mathbf{K}_3\end{bmatrix}\begin{bmatrix}\mathbf{U}_{cIII4p1}\\ \mathbf{U}_{sIII4p1}\end{bmatrix} \tag{56}$$

The stresses based on this displacement field have the form

$$\left\{ \begin{array}{c} \boldsymbol{\sigma}_{c1} \\ \boldsymbol{\sigma}_{s1} \end{array} \right\} = z \left\{ \begin{array}{c} \boldsymbol{\sigma}_{cp1} \\ \boldsymbol{\sigma}_{ps1} \end{array} \right\} + \left\{ \begin{array}{c} \boldsymbol{\sigma}_{cp2} \\ \boldsymbol{\sigma}_{ps2} \end{array} \right\}, \tag{57}$$

where $(\boldsymbol{\sigma}_{pc1}, \boldsymbol{\sigma}_{ps1})$ was given by Eq. (41) and the new term has the form

$$\begin{aligned} \boldsymbol{\sigma}_{cp2} &= \mathbf{C}\left[\boldsymbol{b}_r \mathbf{u}_{cIII4p2} + \boldsymbol{b}_\theta \boldsymbol{u}_{sIII4p2} + \boldsymbol{b}_z \mathbf{u}_{cIII4p1}\right], \\ \boldsymbol{\sigma}_{sp2} &= \mathbf{C}\left[\boldsymbol{b}_r \mathbf{u}_{sIII4p2} - \boldsymbol{b}_\theta \mathbf{u}_{cIII4p2} + \boldsymbol{b}_z \mathbf{u}_{sIII4p1}\right]. \end{aligned} \tag{58}$$

The set of particular solutions $\mathbf{U}_{cIII4p1}$ and $\mathbf{U}_{sIII4p1}$ and corresponding stress field $\boldsymbol{\sigma}_{cp1}$ and $\boldsymbol{\sigma}_{sp1}$ that account for the linearly varying portion of the longitudinal surface shear also yields a transverse force and moment resultants. These resultants must be negated by the superposition of pure and transverse shear force of Problems I and II. Like the other case, equilibria is satisfied on an integral basis, leaving a point-wise self-equilibrated state which decays into the interior.

7. Examples

Two thickness profiles are considered in our examples: (1) a homogeneous, isotropic cylinder with shear modulus G, Poisson's ratio $\nu = 0.3$, and (2) two-layer laminated $\pm 30°$ angle-ply cylinder with equal thickness plies of mechanical properties

$$\frac{E_L}{E_T} = 20, \qquad \frac{G_{LT}}{E_T} = 0.4, \qquad \frac{G_{TT}}{E_T} = 0.3, \qquad \nu_{TT} = 0.3, \qquad \nu_{LT} = 0.2. \tag{59}$$

This laminate profile is the same as that in the previous paper of Huang and Dong (2001), and the C_{ij} properties may be found there. The mean radius/thickness ratio for the homogeneous, isotropic cylinder is $R/H = 1$, and two radius/thickness ratios, $R/H = 1$ and 10, for the angle-ply profile were used to show the differences between thick-walled cylinder and shell behavior.

The κ_{I44}, κ_{II45} and κ_{III44} values for these cylinders are summarized in Table 1 in terms of stiffness G or E_T, thickness H and the R/H ratios with superscripts to identify the type of applied surface traction. In all plots of stress distributions, the stress components are normalized by σ_0 that depends on the various loads as follows: (1) bending moment M_0; $\sigma_0 = M_0/H^3$, (2) transverse shear force P_0; $\sigma_0 = P_0/H^2$, (3) uniform normal and shear tractions σ_i; $\sigma_0 = \sigma_i$, and (4) linearly varying tractions σ_i'; $\sigma_0' = \sigma_0' H$.

Table 1
Stiffness coefficients

	Isotropic	Two layer $\pm 30°$	Two layer $\pm 30°$
	$R/H = 1$	$R/H = 1$	$R/H = 10$
κ_{I44}	10.21 GH^4	16.77 $E_T H^4$	16397 $E_T H^4$
κ_{II45}	0.0	−28.41 $E_T H^5$	−28022 $E_T H^5$
κ_{III4}[a]	15.15 GH^6	−70.43 $E_T H^6$	−3487755 $E_T H^6$
κ_{III4}[b]	28.94 GH^6	6.31 $E_T H^6$	−2384848 $E_T H^6$
κ_{III4}[c]	21.28 GH^6	−72.57 $E_T H^6$	−3562853 $E_T H^6$
κ_{III4}[d]	22.81 GH^6	−51.66 $E_T H^6$	−1820496 $E_T H^6$

[a] Normal pressure on outer surface.
[b] Shear traction on outer surface.
[c] Normal pressure on inner surface.
[d] Shear traction on inner surface.

8. Hollow homogeneous, isotropic cylinder

The stress distributions shown in Fig. 2a–d are, respectively, for the following loading conditions: (1) pure bending and transverse shear, (2) transverse load by surface pressure and hoop shear, (3) uniform longitudinal surface shear, and (4) linearly varying longitudinal shear. Some comments are offered for each of these figures.

Fig. 2a: The pure bending stress is linear over the cross-section. In Timosheko and Goodier (1970, p. 358), is a formula for the shear stress distribution in a solid isotropic cylinder due a transverse shear force. Although not shown here, the present shear stress results are in excellent agreement with this formula.

Fig. 2b: Four applied surface traction for transverse load cases are shown. In all cases, bending stress σ_{zz} with a moment resultant accompanies this loading condition, which is negated by a pure bending moment through c_{III4} and factor $\kappa_{III44}/\kappa_{I44}$ of Eq. (54).

Fig. 2c: For uniform longitudinal surface shears on the inner and outer surfaces, there are shear tractions on the end cross-section with a transverse shear force. Addition of an opposite transverse shear force solution cancels this force on an integral basis but leave a self-equilibrated stress state that decays into the interior.

Fig. 2d: The linearly varying longitudinal shear load incurs a surface stress σ_{rr}. To negate this surface traction, a transverse load problem is needed. Also, σ_{zz} occurs resulting in a bending moment on the cross-

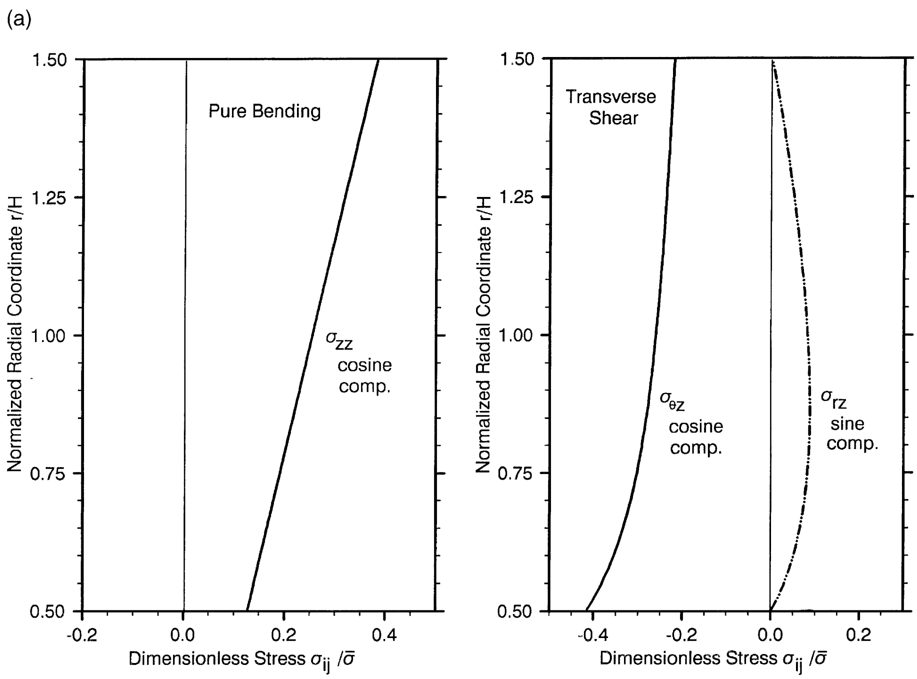

Fig. 2. (a) Normalized stresses for pure bending M_0 and transverse shear force P_0 in homogeneous, isotropic cylinder. (b) Normalized stresses for various surface tractions leading to a transverse load in homogeneous, isotropic cylinder. (c) Normalized stress for uniform longitudinal shears on outer and inner surfaces in homogeneous, isotropic cylinder. (d) Normalized stress for linearly varying longitudinal shears on outer and inner surfaces in homogeneous, isotropic cylinder.

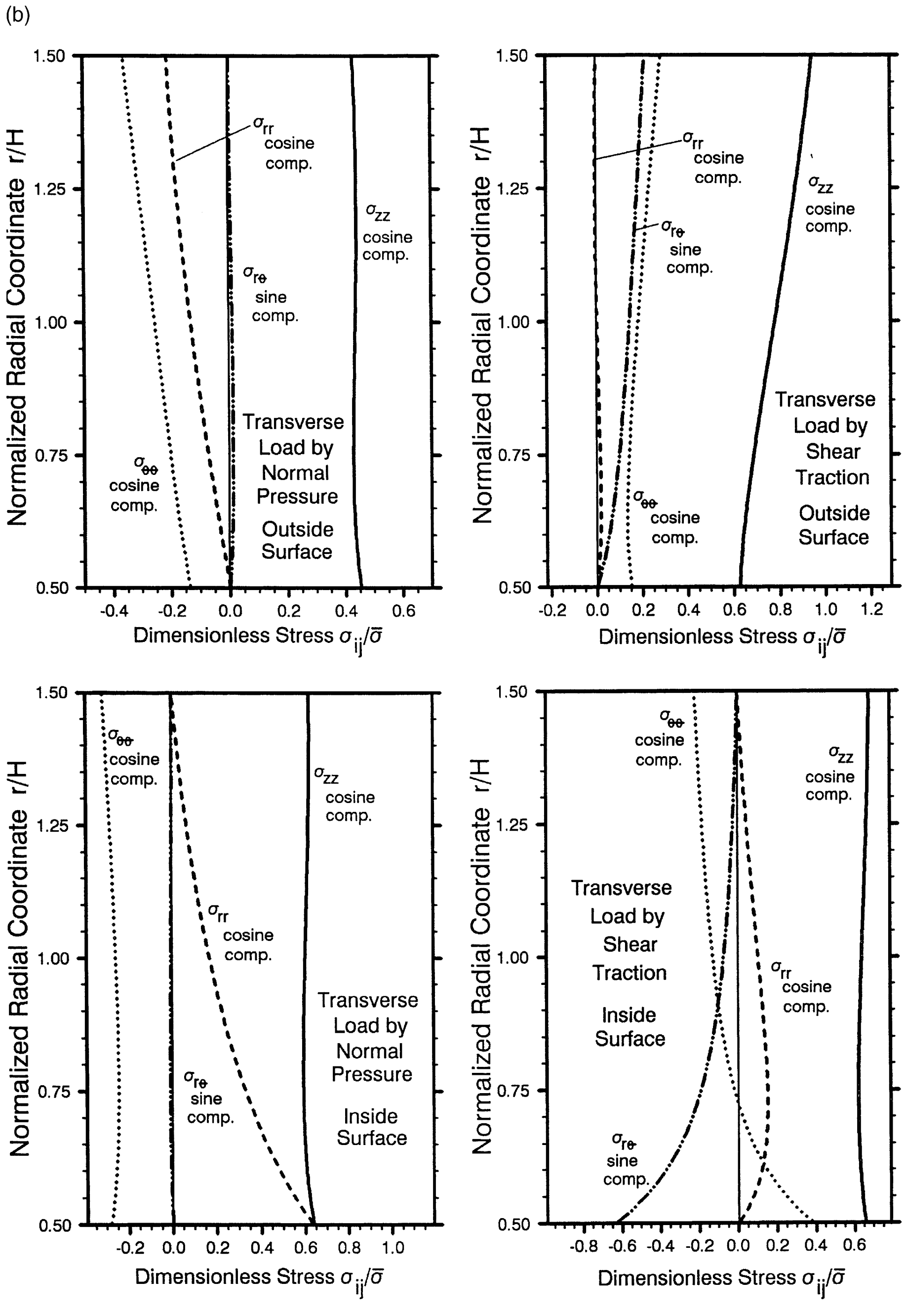

Fig. 2 (*continued*)

section so that a pure bending solution is needed. Again, equilibrium on an integral basis is satisfied, but a self-equilibrated stress state remains. This loading condition on an isotropic cylinder appears to be so deceivingly simple that the authors expected to find it in the literature; however, they were unable to do so.

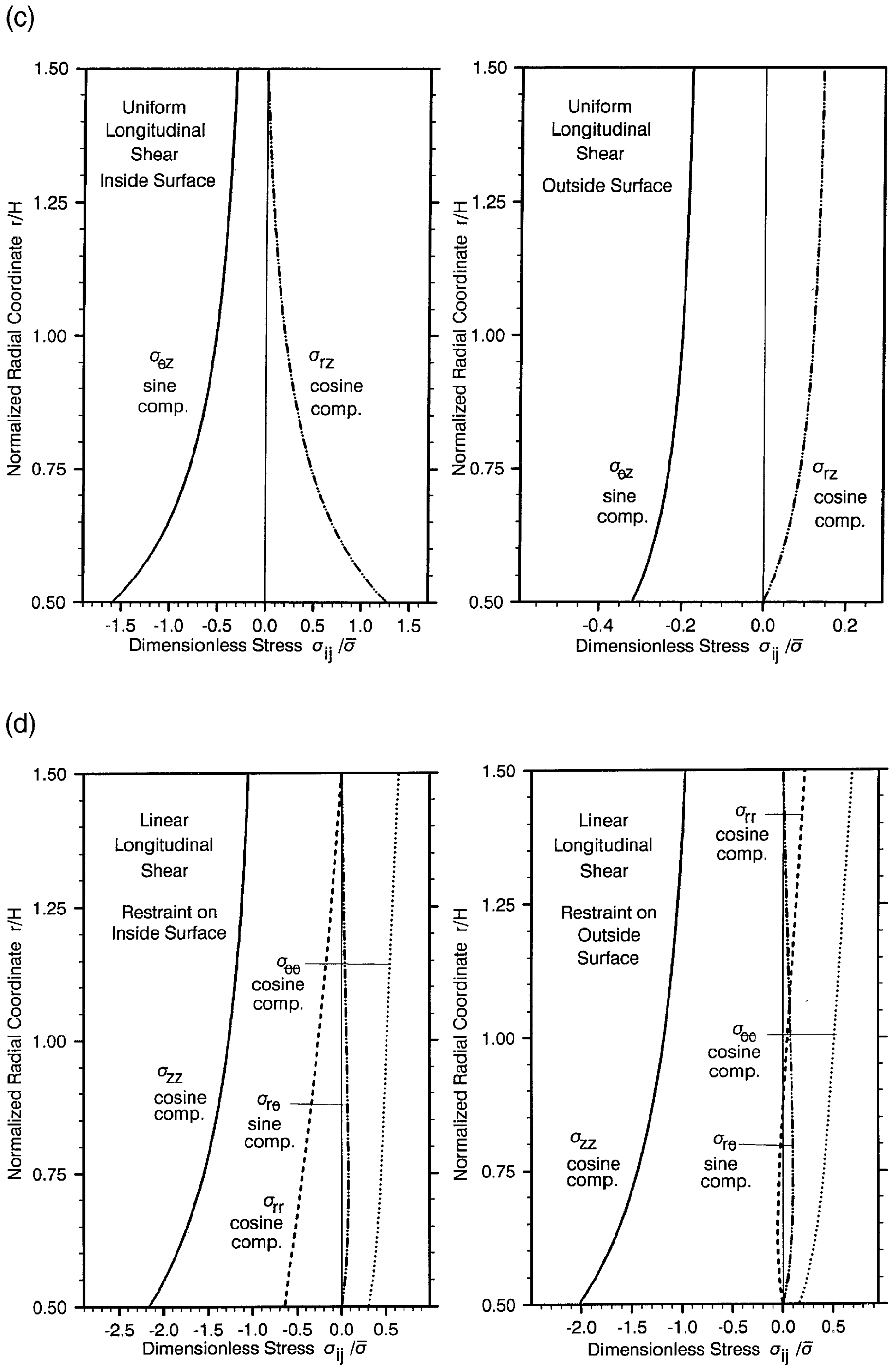

Fig. 2 (*continued*)

9. Two-layer $\pm 30^\circ$ angle-ply cylinder

Five load cases are shown in Fig. 3a–e. They are (1) pure bending, (2) transverse shear force, (3) external pressure leading to a transverse load resultant, (4) external longitudinal shear leading to a uniform moment resultant, and (5) linearly varying external longitudinal shear producing a resultant moment rate. Brief comments are given for each case.

Fig. 3a: For both R/H ratios, a substantial $\sigma_{\theta z}$ component from extension-shear coupling accompanies the σ_{zz} component. For $R/H = 10$, anisotropy and laminate construction have little effect on the bending stress distribution; it is nearly linear except for a very slight interface kink even though C_{33} is the same for both layers. For higher R/H values, this kink will be even less pronounced. All other stress components are virtually zero. Thus, laminated shell and plate theory's ability to predict the stress distributions accurately is validated. But, for thick-walled cylinders such as $R/H = 1$, the bending stress σ_{zz} is no longer linear over the cross-section so that three-dimensional elasticity must be used. Other stress components are present, which are more significant than that for $R/H = 10$.

Fig. 3b: A σ_{zz} accompanies the shear stresses in the $(\sigma_{c1}, \sigma_{s1})$ states, showing that the shear force induces an axially uniform bending moment. This bending moment is negated by stresses associated with the b_{IIi}'s coefficients through κ_{II45} as shown in Eq. (38). This cancellation leaves a self-equilibrated bending stress distribution, whose distribution may be gleaned by comparing the σ_{zz}'s in Fig. 3a and b. In contrast, no such moment occurs in the homogeneous, isotropic cylinder as $\kappa_{II45} = 0$. A linearly varying moment must accompany this transverse force according to global equilibrium whose gradient has a stress distribution for that of pure bending, i.e., according to $z\sigma_{c0}$ and $z\sigma_{s0}$.

Fig. 3c: Plots of stresses for uniform external normal pressure on the cylinder show that a uniform bending moment and transverse shear force are induced by the anisotropy and laminate construction. These resultants are automatically negated by the terms associated with the b_{IIIi}'s and c_{IIIi}'s terms through κ_{II45} and κ_{III44} in Eqs. (53) and (54). The stress distributions for $R/H = 10$ illustrate that they can be predicted by laminated shell theory with reasonable accuracy. This claim is not valid for thick-walled cylinders as seen from the results for $R/H = 1$.

Fig. 3d: Integrating the stress distributions due to external longitudinal shear yields a resultant shear force. To negate this force, a transverse force solution must be appended. This superposition leaves a self-equilibrated state as seen by comparing Fig. 3d with b. While no plots were given for the surface shear on the inner surface, it is remarked that they evince similar distributions as those of traction on the outer surface.

Fig. 3e: Because the exterior surface was restrained for the case of a linearly varying external longitudinal shear, a normal stress σ_{rr} appears on the outer surface for load. This is the reversal of role that was mentioned earlier. To eliminate this normal surface traction, a normal pressure load solution must be appended. In addition to this stress component, σ_{rz}, $\sigma_{\theta z}$, and σ_{zz} occur that lead to a bending moment and transverse shear force. They must be annihilated by superposition of pure bending and transverse shear solutions. The net result of combining all of these load cases is to produce a self-equilibrated stress state that will decay into the interior according to Saint-Venant's Principle.

10. Concluding remarks

A analysis procedure was presented for flexural deformations in laminated anisotropic circular cylinders due to pure bending, transverse shear force, surface tractions with a lateral force resultant, and uniform and linearly varying longitudinal surface shears. Examples for two type of thickness profiles were given to illustrate the behavior. In the example of a two-layer angle-ply-ply laminated anisotropic cylinder, the stress distributions were extraordinary complicated with considerable coupling between longitudinal and shear

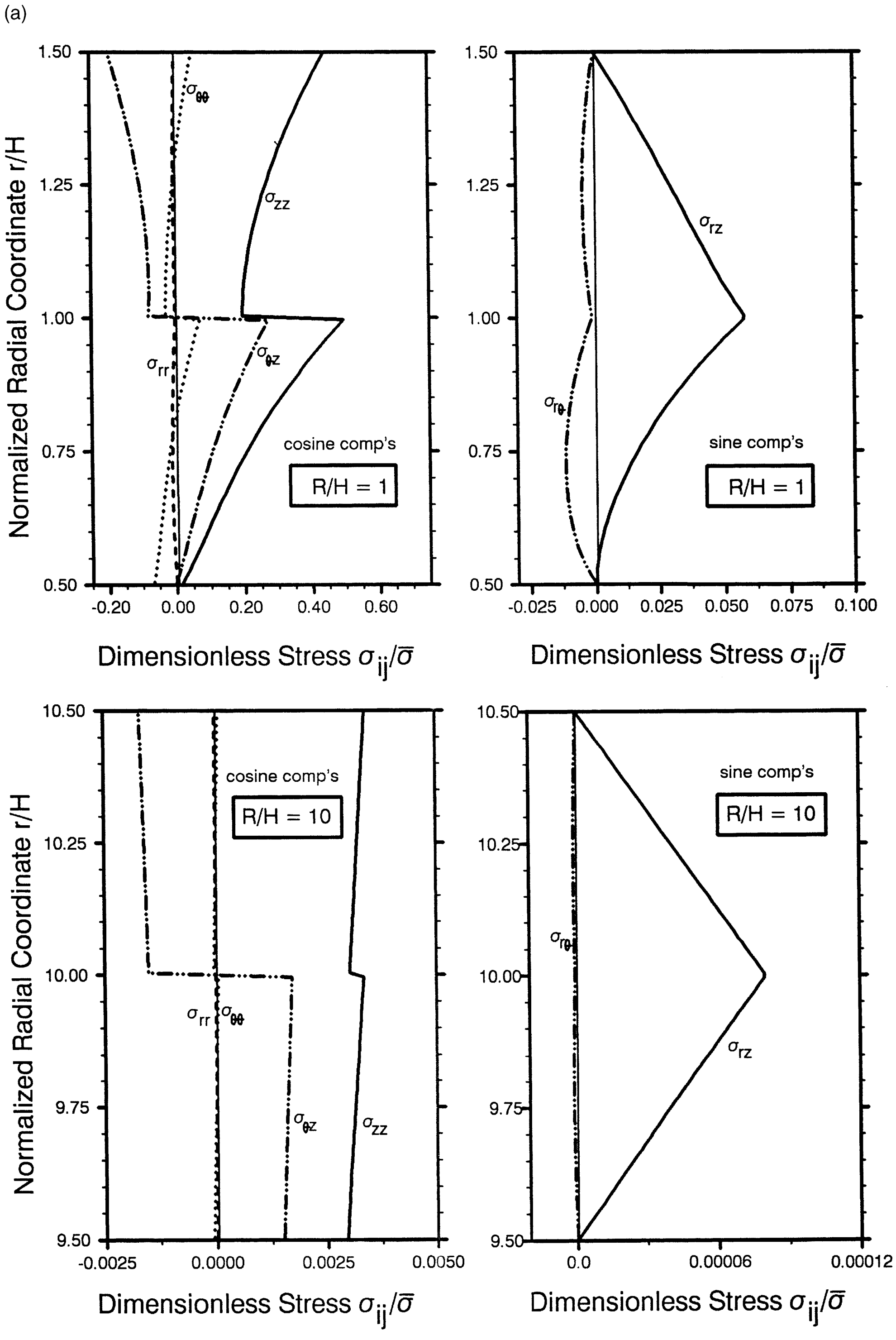

Fig. 3. (a) Normalized stress for pure bending M_0 in a two-layer ±30° angle-ply cylinder. (b) Normalized stress for transverse shear force P_0 in a two-layer ±30° angle-ply cylinder. (c) Normalized stress for normal pressure p_0 on the outer surface of a two-Layer ±30° angle-ply cylinder. (d) Normalized stress for uniform longitudinal shear p_{czo} on the outer surface of a two-layer ±30° angle-ply cylinder. (e) Normalized warpage stress for linearly varying longitudinal shear p'_{czo} on the outer surface of a two-layer ±30° angle-ply cylinder.

(b)

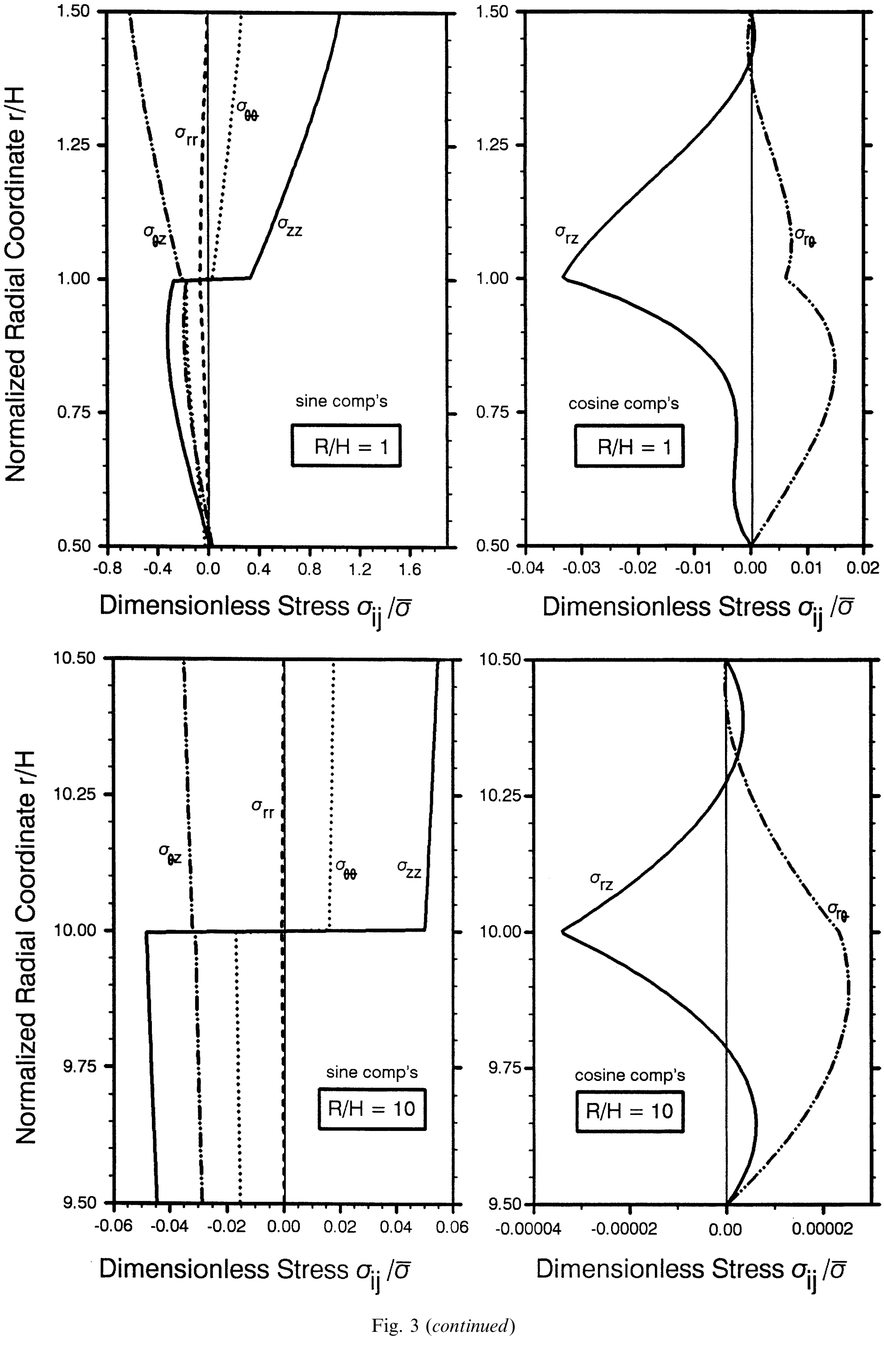

Fig. 3 (*continued*)

(c)

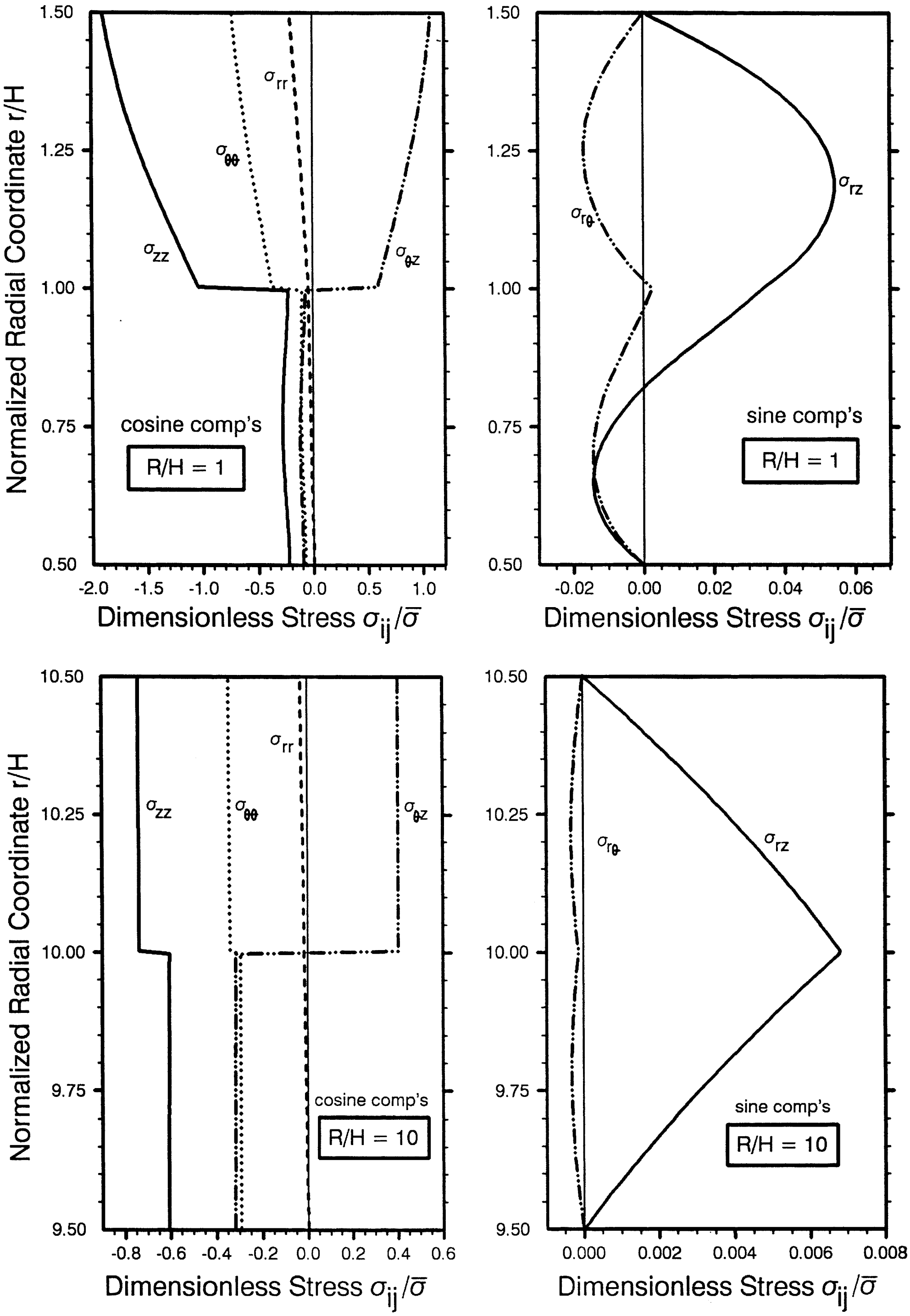

Fig. 3 (*continued*)

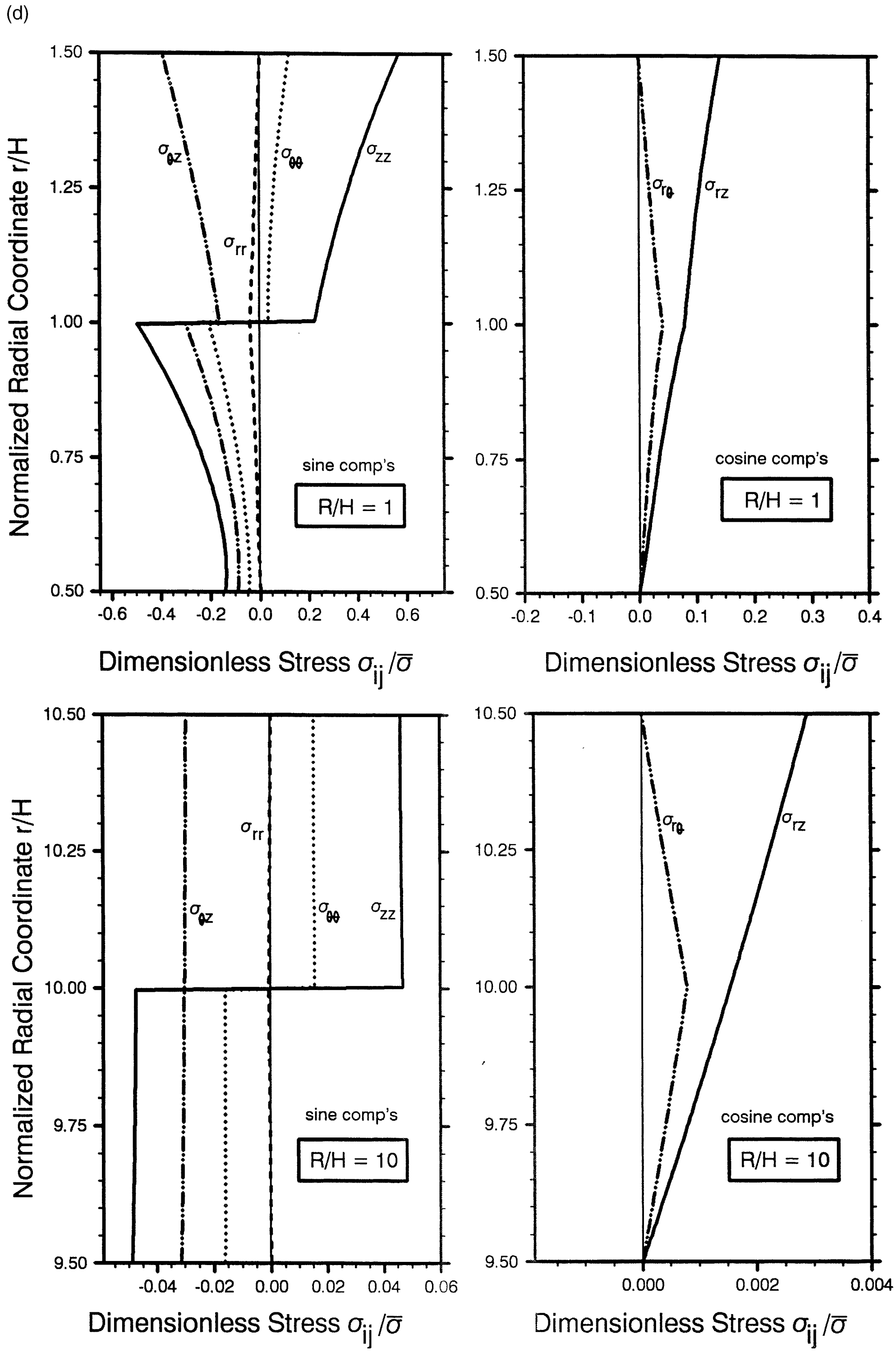

Fig. 3 (*continued*)

(e)

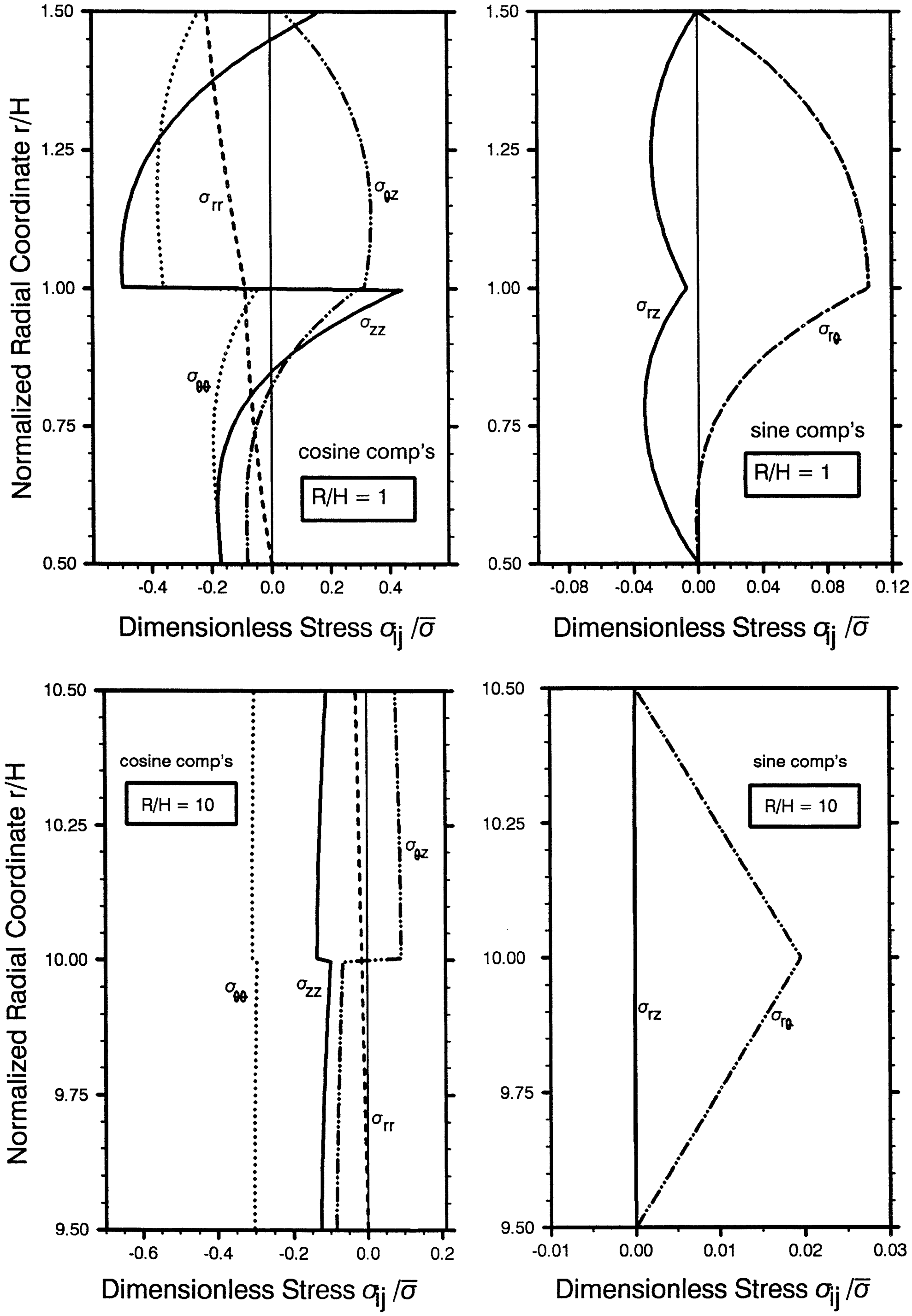

Fig. 3 (*continued*)

deformations. Also, it was observed that in many of these analyses, the solution satisfied the end conditions only on an integral basis rather than on a point-wise basis, leaving self-equilibrated tractions over the cross-section. To render the end conditions free of traction, it is necessary to superpose an end effects analysis.

The end analysis, or the quantification of Saint-Venant's principle, relies on eigendata from the following algebraic eigenproblem

$$\begin{bmatrix} \mathbf{K}_1+\mathbf{K}_4 & \mathbf{K}_2 \\ -\mathbf{K}_2 & \mathbf{K}_1+\mathbf{K}_4 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c0} \\ \mathbf{U}_{s0} \end{Bmatrix} - \gamma \begin{bmatrix} \mathbf{K}_3 & -\mathbf{K}_5 \\ \mathbf{K}_5 & \mathbf{K}_3 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c0} \\ \mathbf{U}_{s0} \end{Bmatrix} - \gamma^2 \begin{bmatrix} \mathbf{K}_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_6 \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{c0} \\ \mathbf{U}_{s0} \end{Bmatrix} = 0, \tag{60}$$

where γ is the inverse decay length. This problem is obtained from the homogeneous form Eq. (6) using the following solution form in it.

$$\begin{Bmatrix} \mathbf{U}_c \\ \mathbf{U}_s \end{Bmatrix} = e^{-\gamma z} \begin{Bmatrix} \mathbf{U}_{c0} \\ \mathbf{U}_{s0} \end{Bmatrix}. \tag{61}$$

Self-equilibrated stress states can be represented in terms of the eigendata of Eq. (60). It is noted that flexural eigendata for a homogeneous, isotropic cylinder were first given by Klemm and Little (1970) from an analytical solution of a boundary-value problem. Some examples of this representation based on finite element formulations may be found in Kazic and Dong (1990) and Lin et al. (2001). Also, the inverse decay length data can be obtained from Zhuang et al.'s (1999) formulation by setting their steady-steady frequency $\omega = 0$. It is mentioned that the bases for this type of analysis are the strain energy decay inequality theorems of Toupin (1965) and Knowles (1966).

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